Professor F. Grillot

Problem set 2

EE 270 - Applied Quantum Mechanics

Due Wednesday Nov. 8, 2017 at 8.00 AM

Exercise I (10 points)
Show that the commutator of two Hermitian operators must be anti-Hermitian. An anti-Hermitian operator $\hat{X}$ obeys $\hat{X}^\dagger = -\hat{X}$. (Hint : $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$ for any two operators $\hat{A}$, $\hat{B}$.)

Exercise II (10 points)
Let $\hat{A}$, $\hat{B}$,and $\hat{C}$ be operators.
(a) Show that $[A, BC] = [A, B]C + B[A, C]$.
(b) Let $\hat{x}$ and $\hat{p}$ be operators satisfying $[\hat{x}, \hat{p}] = i\hbar$ and let the Hamiltonian $\hat{H}$ be $H = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2$. Use commutation relationships and part (a) to find $\frac{d<x>}{dt}$,$\frac{d<p>}{dt}$, and $\frac{d<H>}{dt}$.
(c) Compare your answers with what you expect from the classical situation.

Exercise III (20 points)
Consider a particle confined by the potential $V(x) = 0$ for $0 < x < L$ and $V(x) = \infty$ elsewhere.
(a) Calculate $< x >$, $< x^2 >$, and Δx^2.
(b) Show that as the state number $n \to \infty$ the average values approach those obtained from classical mechanics. Calculate the average particle momentum $< p_x >$, $< p_x^2 >$, and Δp_x^2 as a function of state n. How does $\Delta x \Delta p_x$ depend upon n?

Exercise IV (10 points)
Show that :
(a) The position operator $\hat{x}$ acting on a wave function $\psi(x)$ is Hermitian.
(b) The operator d/dx acting on the wave function $\psi(x)$ is anti-Hermitian.

(c) The momentum operator $-i\hbar(d/dx)$ acting on the wave function $\psi(x)$ is Hermitian.

Exercise V (10 points)
A two-dimensional potential for a particle of mass m is of the form

$$V(x, y) = m\omega^2(x^2 + xy + y^2)$$

Write the potential as a 2×2 matrix and find the new coordinates u and v that diagonalize the matrix. Find the energy levels of the particle.

Exercise VI (10 points)
Using the fact that the Hamiltonian appearing in the Schrödinger equation

$$\frac{-i}{\hbar}\hat{H}\,|\psi(\mathbf{r},\mathbf{t})\rangle = \left|\frac{\partial}{\partial t}\psi(\mathbf{r},\mathbf{t})\right\rangle$$

is Hermitian, show that the time dependence of the average value of the observable A associated with the operator $\hat{A}$ is

$$\frac{d}{dt} < A >= \frac{i}{\hbar} < [\hat{H}, \hat{A}] > + < \frac{\partial}{\partial t}\hat{A} >$$

Exercise VII (20 points)
For the one dimensional harmonic oscillator we have

$$\hat{a} = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i\hat{p}_x}{m\omega}\right)$$

(a) Show that $\langle n|\hat{a}^\dagger|m\rangle = \langle m|\hat{a}|n\rangle^*$ and $\left\langle \hat{a}^\dagger n \middle| \hat{a}^\dagger n \right\rangle = \left\langle n \middle| \hat{a}\hat{a}^\dagger n \right\rangle$
(b) Is $\hat{a}^\dagger$ a Hermitian operator ?
(c) Is $\hat{N} = \hat{a}^\dagger\hat{a}$ a Hermitian operator ?

Exercise VIII (10 points)
Consider a 2-D Hilbert space spanned by an orthonormal basis $|1\rangle, |2\rangle$. We define two kets $|\alpha\rangle = i\,|1\rangle - 2\,|2\rangle$ and $|\beta\rangle = i\,|1\rangle + 2\,|2\rangle$. (Don't worry about normalization in this problem.)
(a) Construct $\langle\alpha|$ and $\langle\beta|$ in terms of the dual basis $\langle 1|$ and $\langle 2|$.
(b) Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$ and confirm that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$
(c) Suppose the operator $\hat{A} = |\alpha\rangle\langle\beta|$. Find the matrix elements of $\hat{A}$ and construct the matrix representation in the basis $|1\rangle, |2\rangle$. Is $\hat{A}$ Hermitian ?